

$K\pi$ vector form factor, dispersive constraints and $\tau \rightarrow \nu_\tau K\pi$ decays

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Abstract: Recent experimental data for the differential decay distribution of the decay $\tau^- \rightarrow \nu_\tau K_S \pi^-$ by the Belle collaboration are described by a theoretical model which is composed of the contributing vector and scalar form factors $F_+^{K\pi}(s)$ and $F_0^{K\pi}(s)$. Both form factors are constructed such that they fulfil constraints posed by analyticity and unitarity. A good description of the experimental measurement is achieved by incorporating two vector resonances and working with a three-times subtracted dispersion relation in order to suppress higher-energy contributions. The resonance parameters of the charged $K^*(892)$ meson, defined as the pole of $F_+^{K\pi}(s)$ in the complex s -plane, can be extracted, with the result $M_{K^*} = 892.0 \pm 0.9 \text{ MeV}$ and $\Gamma_{K^*} = 46.2 \pm 0.4 \text{ MeV}$. Finally, employing a three-subtracted dispersion relation allows to determine the slope and curvature parameters $\lambda'_+ = (24.7 \pm 0.8) \cdot 10^{-3}$ and $\lambda''_+ = (12.0 \pm 0.2) \cdot 10^{-4}$ of the vector form factor $F_+^{K\pi}(s)$ directly from the data.

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1 Introduction

Hadronic decays of the τ lepton provide a fruitful environment to study low-energy QCD under rather clean conditions [1–5]. Many fundamental QCD parameters can be determined from investigations of the τ hadronic width as well as invariant mass distributions. A prime example in this respect is the QCD coupling α_s [6–8]. In addition, fundamental parameters of the strange sector in the Standard Model can also be obtained from the τ strange spectral function. The experimental separation of the Cabibbo-allowed decays and Cabibbo-suppressed modes into strange particles [9–11] paved the way for a new determination of the quark-mixing matrix element $|V_{us}|$ [12–15] as well as the mass of the strange quark [16–23].

Cabibbo-suppressed τ -decays into strange final states are dominated by $\tau \rightarrow \nu_\tau K\pi$. In the past, ALEPH [9] and OPAL [10] have measured the corresponding distribution function but lately B -factories have become a new source of high-statistics data for this reaction. Recently, the Belle experiment published results for the $\tau \rightarrow \nu_\tau K\pi$ spectrum [24] and a new determination of the total branching fraction became available from BaBar [25–27]. In the future, there are good prospects for results on the full spectrum both from BaBar and BESIII.

Theoretically, the general expression for the differential decay distribution of the decay $\tau \rightarrow \nu_\tau K\pi$ can be written as [28]

$$\frac{d\Gamma_{K\pi}}{d\sqrt{s}} = \frac{G_F^2 |V_{us}|^2 M_\tau^3}{32\pi^3 s} S_{\text{EW}} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left[\left(1 + 2 \frac{s}{M_\tau^2}\right) q_{K\pi}^3 |F_+^{K\pi}(s)|^2 + \frac{3\Delta_{K\pi}^2}{4s} q_{K\pi} |F_0^{K\pi}(s)|^2 \right], \quad (1)$$

where isospin invariance is assumed and we have summed over the two possible decay channels $\tau^- \rightarrow \nu_\tau \bar{K}^0 \pi^-$ and $\tau^- \rightarrow \nu_\tau K^- \pi^0$, with the individual decays contributing in the ratio 2 : 1 respectively. Furthermore, $S_{\text{EW}} = 1.0201$ [29] represents an electro-weak correction factor, $\Delta_{K\pi} \equiv M_K^2 - M_\pi^2$, and $q_{K\pi}$ is the kaon momentum in the rest frame of the hadronic system,

$$q_{K\pi}(s) = \frac{1}{2\sqrt{s}} \sqrt{\left(s - (M_K + M_\pi)^2\right) \left(s - (M_K - M_\pi)^2\right)} \cdot \theta\left(s - (M_K + M_\pi)^2\right). \quad (2)$$

Finally, we denote by $F_+^{K\pi}(s)$ and $F_0^{K\pi}(s)$ the vector and scalar $K\pi$ form factors respectively, which we will discuss in detail below.

In eq. (1), the prevailing contribution is due to the $K\pi$ vector form factor $F_+^{K\pi}(s)$, and in the energy region of interest, this form factor is by far dominated by the $K^*(892)$ meson. A description of $F_+^{K\pi}(s)$ based on the chiral theory with resonances (R χ T) [30, 31] was provided in ref. [32], analogous to a similar description of the pion form factor presented in refs. [33–35]. Then in ref. [36] this description was employed in fitting Belle data for the spectrum of the decay $\tau \rightarrow \nu_\tau K_S \pi^-$ [24]. The additionally required scalar $K\pi$ form factor $F_0^{K\pi}(s)$ had been calculated in the same R χ T plus dispersive constraint framework in a series of articles [37–39], and the recent update of $F_0^{K\pi}(s)$ [40] was incorporated as well.

A slight drawback of the description for the vector form factors of refs. [32, 33] is that the form factors only satisfy the analyticity constraints in a perturbative sense, that is up to higher orders in the chiral expansion. Though the violation of analyticity is expected to only be a small correction (of order p^6 in the chiral expansion in the case at hand) it is

certainly worthwhile to corroborate this assumption. A coupled channel analysis of the $K\pi$ vector form factor, which would allow for such a test, was performed in ref. [41]. However, in ref. [41] the theoretical description was not really fitted to the experimental data, so that it is difficult to decide if differences of the coupled channel analysis [41] as compared to the description [32, 36] already show up in the current experimental data. Even more so as the fits performed in ref. [36] provided a satisfactory description of the Belle spectrum.

Below, we shall investigate the related questions in a more modest approach. In the region of the $K^*(892)$ meson, elastic unitarity is still expected to hold. Since this meson dominates the $K\pi$ vector form factor, an ansatz implementing elastic unitarity should result in a good approximation. For the pion vector form factor such an approach was pursued in ref. [34] and in the present work we perform an analogous investigation for $F_+^{K\pi}(s)$. Even though possible coupled-channel contributions are not explicitly included in our parametrisation of the $K\pi$ vector form factor, their influence can be studied through the sensitivity of our ansatz when changing the number of subtractions in the dispersion relation that the form factor satisfies, because a larger number of subtractions entails a stronger suppression of higher-energy contributions. As a benefit of our approach, we are able to extract the resonance parameters of the $K^*(892)$ from the pole of $F_+^{K\pi}(s)$ in the complex s -plane, which should be regarded as more model-independent than the Breit-Wigner type parameters extracted in the previous analyses [24, 36], as well as the first three slopes in the Taylor expansion of $F_+^{K\pi}(s)$ around $s = 0$.

2 The $K\pi$ vector form factor

The $K\pi$ vector form factor $F_+^{K\pi}(s)$ is an analytic function in the complex s -plane, except for a cut along the positive real axis, starting at the $K\pi$ threshold $s_{K\pi} \equiv (M_K + M_\pi)^2$, where its imaginary part develops a discontinuity. The analyticity and unitarity properties of the form factor result in the fact that it satisfies an n -subtracted dispersion relation, explicated in more detail for example in refs. [33, 34]. In the elastic region below roughly 1.2 GeV, the dispersion relation admits the well-known Omnès solution [42]

$$F_+^{K\pi}(s) = P_n(s) \exp \left\{ \frac{s^n}{\pi} \int_{s_{K\pi}}^{\infty} ds' \frac{\delta_1^{K\pi}(s')}{(s')^n (s' - s - i0)} \right\}, \quad (3)$$

which corresponds to performing the n subtractions at $s = 0$, and where

$$P_n(s) = \exp \left\{ \sum_{k=0}^{n-1} \frac{s^k}{k!} \frac{d^k}{ds^k} \ln F_+^{K\pi}(s) \Big|_{s=0} \right\} \quad (4)$$

is the subtraction polynomial. More general formulae with subtractions at an arbitrary point $s = s_0$ can for example be found in ref. [43]. Furthermore, $\delta_1^{K\pi}(s)$ is the P-wave $I = 1/2$ elastic $K\pi$ phase shift. As on general grounds the phase shift is expected to go to an integer multiple of π , at least one subtraction ($n = 1$) is required in eq. (3) to make the integral convergent. Let us first dwell on this case in more detail, before turning to the case with a larger number of subtractions.

While in the approach of refs. [32, 33] to the vector form factor the real part of the one-loop integral function $\tilde{H}(s)$ (for its definition see [32]) is resummed into an exponential, strict unitarity is only maintained if this piece, together with the imaginary part which provides the width of the resonance, is resummed in the denominator of the form factor [37, 44]. The resulting expression of the vector form factor corresponding to one single resonance then reads

$$F_+^{K\pi}(s) = \frac{m_{K^*}^2}{m_{K^*}^2 - s - \kappa \tilde{H}_{K\pi}(s)}. \quad (5)$$

Of course, up to order p^4 in the chiral expansion, resumming the real part in the denominator or an exponential is fully equivalent. Differences of both approaches first start to appear at $\mathcal{O}(p^6)$. In eq. (5) the parameter m_{K^*} is to be distinguished from the true mass of the K^* meson M_{K^*} , which later will be identified with the real part of the pole position of $F_+^{K\pi}(s)$ in the complex s -plane.¹

Identifying the imaginary part in the denominator of eq. (5) with $-m_{K^*}\gamma_{K^*}(s)$, where the s -dependent width of the K^* meson takes the generic form of a vector resonance,

$$\gamma_{K^*}(s) = \gamma_{K^*} \frac{s}{m_{K^*}^2} \frac{\sigma_{K\pi}^3(s)}{\sigma_{K\pi}^3(m_{K^*}^2)}, \quad (6)$$

with $\gamma_{K^*} \equiv \gamma_{K^*}(m_{K^*}^2)$, the dimensionful constant κ has to take the value:

$$\kappa = \frac{192\pi F_K F_\pi}{\sigma_{K\pi}(m_{K^*}^2)^3} \frac{\gamma_{K^*}}{m_{K^*}}. \quad (7)$$

In eqs. (6) and (7), the phase space function $\sigma_{K\pi}(s)$ is given by $\sigma_{K\pi}(s) = 2q_{K\pi}(s)/\sqrt{s}$. The form factor $F_+^{K\pi}(s)$ can therefore be written in the equivalent form

$$F_+^{K\pi}(s) = \frac{m_{K^*}^2}{m_{K^*}^2 - s - \kappa \text{Re}\tilde{H}_{K\pi}(s) - i m_{K^*}\gamma_{K^*}(s)}. \quad (8)$$

From eq. (8) the normalisation $F_+^{K\pi}(0)$ at $s = 0$ which is needed in order to calculate the reduced form factor $\tilde{F}_+^{K\pi}(s) \equiv F_+^{K\pi}(s)/F_+^{K\pi}(0)$, is given by

$$F_+^{K\pi}(0) = \frac{m_{K^*}^2}{m_{K^*}^2 - \kappa \tilde{H}_{K\pi}(0)}. \quad (9)$$

The phase $\delta_1^{K\pi}(s)$ of the form factor $F_+^{K\pi}(s)$ is found to be:

$$\tan \delta_1^{K\pi}(s) \equiv \frac{\text{Im}F_+^{K\pi}(s)}{\text{Re}F_+^{K\pi}(s)} = \frac{m_{K^*}\gamma_{K^*}(s)}{m_{K^*}^2 - s - \kappa \text{Re}\tilde{H}_{K\pi}(s)}. \quad (10)$$

It is a straightforward matter to verify that with the phase $\delta_1^{K\pi}(s)$ of eq. (10), the form factor of eq. (8) satisfies the Omnès relation (3) with $n = 1$ and $P_1(s) = F_+^{K\pi}(0)$. Therefore, it is

¹The renormalisation scale μ appearing in $\tilde{H}_{K\pi}(s)$ will be set to the physical resonance mass $\mu = M_{K^*}$.

in accord with the analyticity and unitarity requirements. Finally, the reduced form factor is written as

$$\tilde{F}_+^{K\pi}(s) = \frac{m_{K^*}^2 - \kappa \tilde{H}_{K\pi}(0)}{m_{K^*}^2 - s - \kappa \operatorname{Re} \tilde{H}_{K\pi}(s) - i m_{K^*} \gamma_{K^*}(s)}, \quad (11)$$

which resembles the non-strange form factor of Gounaris and Sakurai [45]. (For comparison, see also ref. [46].) However, while in the Breit-Wigner resonance shape of ref. [45] the real part of the inverse propagator is obtained through a twice subtracted dispersion relation combined with proper subtractions to fix mass and width of the resonance, the form factor in eq. (11) is found from a once subtracted dispersion relation satisfying analyticity and unitarity.

As has already been discussed in ref. [36], in eq. (1) only the reduced form factor $\tilde{F}_+^{K\pi}(s)$ has to be modelled, as the normalisation of $F_+^{K\pi}(s)$ only appears in the product $|V_{us}|F_+^{K\pi}(0)$. This combination is determined most precisely from the analysis of semi-leptonic kaon decays. A recent average was presented by the FLAVIANet kaon working group, and reads [47]

$$|V_{us}|F_+^{K^0\pi^-}(0) = 0.21664 \pm 0.00048. \quad (12)$$

In what follows, we work with form factors normalised to one at the origin and assume the value (12) for the overall normalisation. We remark that the normalisation for the scalar and the vector form factors is the same and that (12) already corresponds to the $K^0\pi^-$ channel, which was analysed by the Belle collaboration [24]. Consequently, possible isospin-breaking corrections to $F_+^{K^0\pi^-}(0)$ are properly taken into account.

3 Single resonance fits to the Belle spectrum

Our fits to the Belle $\tau^- \rightarrow \nu_\tau K_S \pi^-$ spectrum [24] will be performed in complete analogy to the recent analysis of ref. [36]. Let us briefly review the main strategy for these fits. The central fit function is taken to have the form

$$\frac{1}{2} \cdot \frac{2}{3} \cdot 0.0115 [\text{GeV}/\text{bin}] \mathcal{N}_T \cdot \frac{1}{\Gamma_\tau \bar{B}_{K\pi}} \frac{d\Gamma_{K\pi}}{d\sqrt{s}}. \quad (13)$$

The factors $1/2$ and $2/3$ arise because the $K_S \pi^-$ channel has been analysed. Then, 11.5 MeV was the bin-width chosen by the Belle collaboration, and $\mathcal{N}_T = 53110$ is the total number of observed signal events. Finally, Γ_τ denotes the total decay width of the τ lepton and $\bar{B}_{K\pi}$ a remaining normalisation factor that will be deduced from the fits. The normalisation of our ansatz (13) is taken such that for a perfect agreement between data and fit function, $\bar{B}_{K\pi}$ would correspond to the total branching fraction $B_{K\pi} \equiv B[\tau^- \rightarrow \nu_\tau K_S \pi^-]$ which is obtained by integrating the decay spectrum. All further numerical input parameters have been chosen as in ref. [36].

Numerically, our first fit of the Belle data [24] with the vector form factor $F_+^{K\pi}(s)$ according to eq. (8), and including data up to $\sqrt{s} = 1.01525 \text{ GeV}$ (centre of bin 34), is presented in table 1. As discussed in ref. [36], we have removed the problematic data points 5, 6 and 7 from the fit. Furthermore, we have not taken into account the lowest data point since for our physical meson masses the centre of this bin lies below the $K\pi$ threshold. For the scalar form factor $F_0^{K\pi}(s)$ we have used the recent update [40], employing the central parameters given

Eq. (8) for $F_+^{K\pi}(s)$	
$\bar{B}_{K\pi} (B_{K\pi})$	$0.3611 \pm 0.0042 \% (0.3562 \%)$
m_{K^*}	$943.35 \pm 0.51 \text{ MeV}$
γ_{K^*}	$66.29 \pm 0.79 \text{ MeV}$
$\chi^2/\text{n.d.f.}$	$39.4/27$

Table 1: Results for the fit with a one-subtracted dispersion relation including a single vector resonance in $F_+^{K\pi}(s)$ according to eq. (8), as well as the scalar form factor $F_0^{K\pi}(s)$ [40].

there. One observes that due to the not very satisfactory quality of the fit, the fit parameter $\bar{B}_{K\pi}$ and the integrated branching fraction $B_{K\pi}$ display a marked deviation, which is however within the uncertainties. Furthermore, because of the real part of the loop-integral in the denominator, m_{K^*} and γ_{K^*} turn out rather different from their physical values M_{K^*} and Γ_{K^*} . For the final results of the parameters in our description of the vector form factor, in our conclusion we shall also present values for the physical parameters M_{K^*} and Γ_{K^*} , as obtained from the pole position in the complex s -plane.

In order to make the fit less sensitive to deficiencies of our description in the higher-energy region, a larger number of subtractions can be applied to the dispersion relation. Besides, employing an n -subtracted dispersion relation has the advantage that the slope parameters which appear as subtraction constants are determined more directly from the data. It should be pointed out, however, that the form factor with a larger number of subtractions $n \geq 2$ violates the expected QCD large-energy behaviour. For large enough energies the form factor $F_+^{K\pi}(s)$ should vanish as $1/s$. Since we only employ the vector form factor up to about $\sqrt{s} \approx 1.7 \text{ GeV}$, which is still in the resonance region, we consider this deficiency acceptable. Anyhow, as can be verified explicitly from our fits, in the considered region above the second vector resonance, $F_+^{K\pi}(s)$ is a decreasing function of s . On the other hand, fits with only one subtraction were generally found to only provide a poor description of the experimental data.

Let us present our results for the case $n = 3$ in detail, but below, we shall also briefly comment on the cases $n = 2$ and $n = 4$. The importance of the high-energy region can be studied by introducing a cutoff s_{cut} as the upper limit of the integration in the Omnès integral (3). Incorporating three subtractions, the reduced form factor $\tilde{F}_+^{K\pi}(s)$ then takes the form:

$$\tilde{F}_+^{K\pi}(s) = \exp \left\{ \alpha_1 \frac{s}{M_{\pi^-}^2} + \frac{1}{2} \alpha_2 \frac{s^2}{M_{\pi^-}^4} + \frac{s^3}{\pi} \int_{s_{K\pi}}^{s_{\text{cut}}} ds' \frac{\delta_1^{K\pi}(s')}{(s')^3 (s' - s - i0)} \right\}, \quad (14)$$

where the phase $\delta_1^{K\pi}(s)$ corresponds to the expression of eq. (10). The parameters α_1 and α_2 can easily be related to the slope parameters $\lambda_+^{(n)}$, which appear in the Taylor expansion of $\tilde{F}_+^{K\pi}(s)$ around $s = 0$:

$$\tilde{F}_+^{K\pi}(s) = 1 + \lambda'_+ \frac{s}{M_{\pi^-}^2} + \frac{1}{2} \lambda''_+ \frac{s^2}{M_{\pi^-}^4} + \frac{1}{6} \lambda'''_+ \frac{s^3}{M_{\pi^-}^6} + \dots \quad (15)$$

Explicitly, the relations for the linear and quadratic slope parameters λ'_+ and λ''_+ then take the form:

$$\lambda'_+ = \alpha_1, \quad \lambda''_+ = \alpha_2 + \alpha_1^2. \quad (16)$$

Below, we shall also compute the cubic slope parameter λ'''_+ from the dispersive integral.

	$s_{\text{cut}} = 3.24 \text{ GeV}^2$	$s_{\text{cut}} = 4 \text{ GeV}^2$	$s_{\text{cut}} = 9 \text{ GeV}^2$	$s_{\text{cut}} \rightarrow \infty$
$\bar{B}_{K\pi}$	$0.394 \pm 0.045 \%$	$0.397 \pm 0.046 \%$	$0.398 \pm 0.046 \%$	$0.398 \pm 0.046 \%$
$(B_{K\pi})$	(0.389%)	(0.391%)	(0.393%)	(0.393%)
$m_{K^*} [\text{MeV}]$	943.34 ± 0.57	943.36 ± 0.58	943.37 ± 0.58	943.37 ± 0.58
$\gamma_{K^*} [\text{MeV}]$	66.48 ± 0.87	66.50 ± 0.89	66.52 ± 0.89	66.52 ± 0.89
$\lambda'_+ \times 10^3$	23.9 ± 2.4	24.2 ± 2.5	24.4 ± 2.5	24.4 ± 2.5
$\lambda''_+ \times 10^4$	11.5 ± 0.7	11.5 ± 0.7	11.5 ± 0.7	11.5 ± 0.7
$\chi^2/\text{n.d.f.}$	45.8/41	45.8/41	45.7/41	45.8/41

Table 2: Results for the fits with a three-subtracted dispersion relation including a single vector resonance in $F_+^{K\pi}(s)$ according to eq. (8), as well as the scalar form factor $F_0^{K\pi}(s)$ [40].

The results of our fits with the three-subtracted dispersion relation, employing four values of $\sqrt{s_{\text{cut}}}$, namely 1.8 GeV, 2 GeV, 3 GeV and $s_{\text{cut}} \rightarrow \infty$, are given in table 2. For these fits, we have included experimental data up to the data point 50 at $\sqrt{s} = 1.19925 \text{ GeV}$ (centre of the bin) [24], and as already mentioned above, removing the problematic data points 5, 6 and 7. As can be seen from table 2, the fits are indeed rather insensitive to the upper integration limit s_{cut} , implying that the higher-energy region is well suppressed. Compared with the fit of table 1, which only employed a single subtraction, with $\chi^2/\text{n.d.f.} \approx 1.1$ the fit quality is substantially improved. Related to that, also the results for $\bar{B}_{K\pi}$ and $B_{K\pi}$ turn out much closer. However, the slope parameter λ'_+ is not well determined from our fit. This originates in the fact that the slope parameters are almost 100 % correlated with the total branching fraction $\bar{B}_{K\pi}$, and this parameter has relatively large uncertainties. Therefore, in our best estimate of the model parameters below, we shall impose the experimental measurement of the total branching fraction $B_{K\pi}$.

Though we only present explicit results for the three-subtracted dispersion relation, we have also investigated the cases $n = 2$ and $n = 4$. In the case $n = 2$, a still somewhat stronger dependence on the cutoff s_{cut} is observed, which is why we do not discuss the corresponding results in detail. The four-subtracted dispersion relation, on the other hand, yields almost unchanged central values for the fit parameters without any improvement in the $\chi^2/\text{n.d.f.}$, but with larger parameter errors due to the additional degree of freedom. From this we conclude that the case $n = 3$ discussed above is an optimal choice as far as the number of subtractions is concerned. Next, we shall also include a second vector resonance, the $K^*(1410)$ into our description of the $K\pi$ vector form factor.

4 Fits with two vector resonances

For our final fits, we aim at a description of the $\tau \rightarrow \nu_\tau K\pi$ spectrum in the full energy range up to the τ mass. To this end, we also introduce a second vector resonance, the $K^{*'} = K^*(1410)$, into our model for the vector form factor $F_+^{K\pi}(s)$. As it is unclear how to directly incorporate a second vector resonance in the phase $\delta_1^{K\pi}(s)$, we have followed a somewhat indirect approach, which should be sufficient for our purposes, as the contribution of the $K^{*'}$ resonance is suppressed by phase space.

Like for the case of the single resonance, as a starting point we assume the form of $F_+^{K\pi}(s)$ given in eq. (5) of ref. [36], however resumming the real part of $\tilde{H}_{K\pi}(s)$ in the denominator of the form factor. This leads to the following expression:

$$\tilde{F}_+^{K\pi}(s) = \frac{m_{K^*}^2 - \kappa_{K^*} \tilde{H}_{K\pi}(0) + \gamma s}{D(m_{K^*}, \gamma_{K^*})} - \frac{\gamma s}{D(m_{K^{*'}}, \gamma_{K^{*'}})}, \quad (17)$$

where

$$D(m_n, \gamma_n) \equiv m_n^2 - s - \kappa_n \text{Re} \tilde{H}_{K\pi}(s) - i m_n \gamma_n(s). \quad (18)$$

For both resonances, $\gamma_n(s)$ is given equivalently to the form of eq. (6), and the corresponding κ_n can be deduced in analogy to eq. (7). Like in eq. (10), the phase $\delta_1^{K\pi}(s)$ can be calculated from the relation

$$\tan \delta_1^{K\pi}(s) = \frac{\text{Im} F_+^{K\pi}(s)}{\text{Re} F_+^{K\pi}(s)}. \quad (19)$$

This is the phase that we then employ in the Omnès integral representation (3) for the form factor to perform our fits.

	$s_{\text{cut}} = 3.24 \text{ GeV}^2$	$s_{\text{cut}} = 4 \text{ GeV}^2$	$s_{\text{cut}} = 9 \text{ GeV}^2$	$s_{\text{cut}} \rightarrow \infty$
$\bar{B}_{K\pi}$	$0.386 \pm 0.043\%$	$0.404 \pm 0.044\%$	$0.417 \pm 0.046\%$	$0.417 \pm 0.046\%$
$(B_{K\pi})$	(0.384%)	(0.402%)	(0.414%)	(0.414%)
$m_{K^*} [\text{MeV}]$	943.27 ± 0.58	943.40 ± 0.57	943.48 ± 0.57	943.49 ± 0.57
$\gamma_{K^*} [\text{MeV}]$	66.43 ± 0.90	66.66 ± 0.87	66.81 ± 0.86	66.81 ± 0.87
$m_{K^{*'}} [\text{MeV}]$	1392 ± 41	1369 ± 30	1361 ± 28	1361 ± 28
$\gamma_{K^{*'}} [\text{MeV}]$	273 ± 137	224 ± 101	212 ± 93	212 ± 93
$\gamma \times 10^2$	-4.2 ± 2.0	-3.6 ± 1.6	-3.4 ± 1.5	-3.4 ± 1.5
$\lambda'_+ \times 10^3$	22.6 ± 2.2	23.9 ± 2.1	24.7 ± 2.1	24.8 ± 2.1
$\lambda''_+ \times 10^4$	11.5 ± 0.6	11.7 ± 0.7	11.9 ± 0.7	11.9 ± 0.7
$\chi^2/\text{n.d.f.}$	73.7/78	75.6/78	77.2/78	77.3/78

Table 3: Results for the fits with a three-subtracted dispersion relation including two vector resonances in $F_+^{K\pi}(s)$ according to eqs. (17) to (19), as well as the scalar form factor $F_0^{K\pi}(s)$ of ref. [40].

Our first fit with two vector resonances proceeds in complete analogy to the second fit in the single-resonance case. Again, we use the three-subtracted dispersion relation, and

investigate four values of s_{cut} , in order to study the dependence of our fits on the higher-energy contributions. Now, the Belle data [24] have been included up to $\sqrt{s} = 1.65925$ GeV (data point 90).² The corresponding fit results are presented in table 3.

The general picture of our fits with two resonances is quite satisfying. The theoretical model provides a good description of the experimental data in the full energy range. The $\chi^2/\text{n.d.f.}$ for all values of s_{cut} turns out smaller than one. The dependence of the resulting fit parameters on s_{cut} is small and within the uncertainties, though clearly visible. The fits provide a precise determination of the parameters of the lowest lying K^* vector resonance, and a still reasonable accuracy for the second $K^{*'}$ resonance. However, the fit uncertainties for the branching fraction $\bar{B}_{K\pi}$ are relatively large. Accordingly, also the error on the slope parameter λ'_+ is found large, because these two parameters are almost 100 % correlated.

	$s_{\text{cut}} = 3.24 \text{ GeV}^2$	$s_{\text{cut}} = 4 \text{ GeV}^2$	$s_{\text{cut}} = 9 \text{ GeV}^2$	$s_{\text{cut}} \rightarrow \infty$
$m_{K^*} [\text{MeV}]$	943.32 ± 0.59	943.41 ± 0.58	943.48 ± 0.57	943.49 ± 0.57
$\gamma_{K^*} [\text{MeV}]$	66.61 ± 0.88	66.72 ± 0.86	66.82 ± 0.85	66.82 ± 0.85
$m_{K^{*'}} [\text{MeV}]$	1407 ± 44	1374 ± 30	1362 ± 26	1362 ± 26
$\gamma_{K^{*'}} [\text{MeV}]$	325 ± 149	240 ± 100	216 ± 86	215 ± 86
$\gamma \times 10^2$	-5.2 ± 2.0	-3.9 ± 1.5	-3.5 ± 1.3	-3.5 ± 1.3
$\lambda'_+ \times 10^3$	24.31 ± 0.74	24.66 ± 0.69	24.94 ± 0.68	24.96 ± 0.67
$\lambda''_+ \times 10^4$	12.04 ± 0.20	11.99 ± 0.19	11.96 ± 0.19	11.96 ± 0.19
$\chi^2/\text{n.d.f.}$	74.2/79	75.7/79	77.2/79	77.3/79

Table 4: Results for the fits with a three-subtracted dispersion relation including two vector resonances in $F_+^{K\pi}(s)$ according to eqs. (17) to (19), as well as the scalar form factor $F_0^{K\pi}(s)$ [40]. The total branching fraction $B_{K\pi}$ has been fixed to the experimental value (20), and the corresponding uncertainty is included in quadrature.

In order to provide a better determination of the slope parameters, a possible way to proceed is to fix the total branching fraction $B_{K\pi}$ to the experimental measurement. A very recent update of the world average has been presented in ref. [24, 26, 27], with the finding:

$$B[\tau^- \rightarrow \nu_\tau K_S \pi^-] = 0.418 \pm 0.011 \% . \quad (20)$$

Thus, for our final fits, we have fixed $B_{K\pi}$ to take this value. The resulting fit parameters for the remaining quantities are displayed in table 4. Here, the quoted uncertainties include the statistical fit errors as well as a variation of $B_{K\pi}$ in the range given by (20). From table 4 we infer that due to the reduction in the uncertainty of $B_{K\pi}$, correspondingly also the uncertainty in the slope parameters is much reduced, while the errors of the remaining parameters to a good approximation stay as before. Also the $\chi^2/\text{n.d.f.}$ is practically unchanged, still remaining below one for all values of s_{cut} .

²Although the full Belle data set consists of 100 data points, we follow a suggestion of the experimentalists to only fit the data up to point 90 [48].

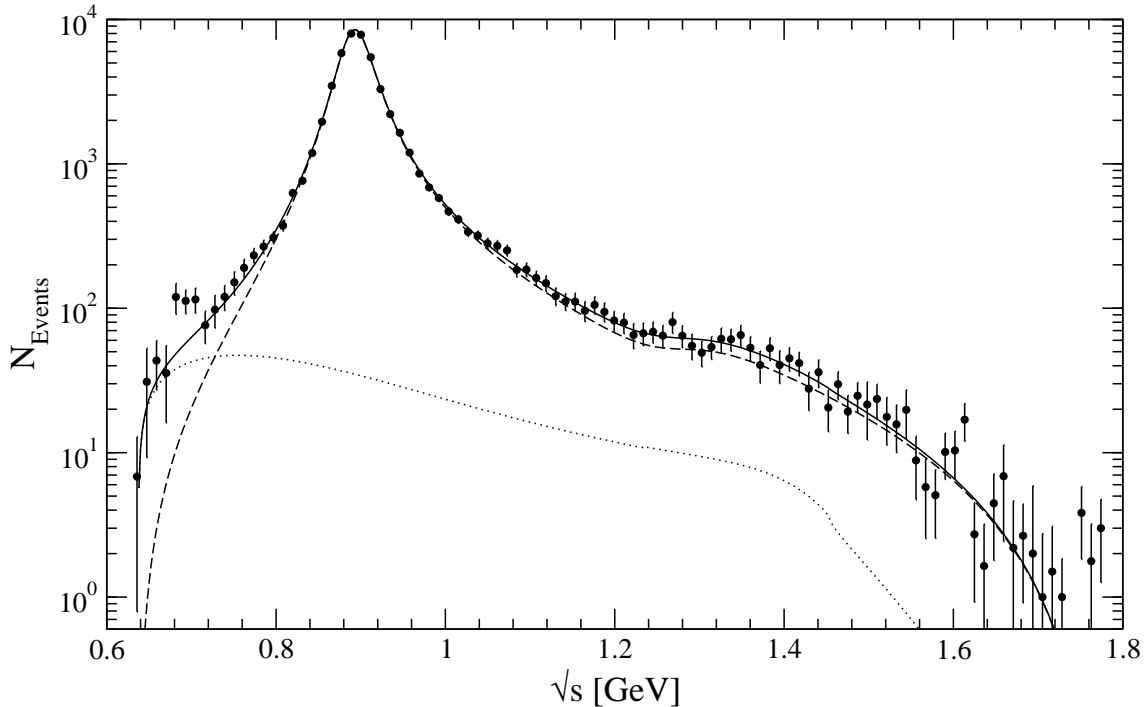


Figure 1: Main fit result to the Belle data [24] for the differential decay distribution of the decay $\tau^- \rightarrow \nu_\tau K_S \pi^-$. Our theoretical description corresponds to the fit of table 4 with $s_{\text{cut}} = 4 \text{ GeV}^2$. The full fit including vector form factor $F_+^{K\pi}(s)$ and scalar form factor $F_0^{K\pi}(s)$ is displayed as the solid line. The separate vector and scalar contributions are shown as the dashed and dotted lines respectively.

A graphical account of our central result is displayed in figure 1. The solid line corresponds to the fit of table 4 including vector form factor $F_+^{K\pi}(s)$ as well as the scalar form factor $F_0^{K\pi}(s)$ at $s_{\text{cut}} = 4 \text{ GeV}^2$. The separate contributions of $F_+^{K\pi}(s)$ and $F_0^{K\pi}(s)$ are shown as the dashed and dotted lines respectively. As is apparent, apart from the data points 5, 6 and 7 in the low-energy region, our model provides a perfect description of the experimental data by the Belle collaboration. Also, from an inspection of the region below the K^* resonance it is evident that a contribution from the scalar form factor $F_0^{K\pi}(s)$ is required, though, like in the analysis of ref. [36], the sensitivity to $F_0^{K\pi}(s)$ is not strong enough to allow for a determination of the corresponding model parameters.

5 Conclusions

Hadronic τ decays provide a means to obtain information on low-energy QCD as well as hadron phenomenology. In this work, we have studied recent data on the decay channel $\tau^- \rightarrow \nu_\tau K_S \pi^-$ by the Belle collaboration [24]. The measured decay spectrum allows to test models for the vector and scalar $K\pi$ form factors $F_+^{K\pi}(s)$ and $F_0^{K\pi}(s)$, and to deduce

the corresponding model parameters for the vector form factor. For $F_+^{K\pi}(s)$, we have used a model which incorporates the constraints on the form factor from analyticity and elastic unitarity. Furthermore, we investigated n -subtracted dispersive integrals with a cutoff s_{cut} , where n ranges from 1 to 4 and $\sqrt{s_{\text{cut}}}$ was varied between 1.8 GeV and infinity. This allowed to test the sensitivity (or insensitivity) of the form factor model to contributions from higher energies which are not well known. For the scalar form factor $F_0^{K\pi}(s)$, we have employed the description of ref. [38], which is based on solving dispersion relations for a two-body coupled-channel problem, and was recently updated in [40].

Let us begin with summarising our final results for the parameters of the K^* and $K^{*'}$ vector resonances. As our central results, we quote the values of table 4 at $s_{\text{cut}} = 4 \text{ GeV}^2$. To the uncertainty given in table 4, we add an error for the variation of our results when changing s_{cut} . The resonance mass and width parameters are then found to be:

$$m_{K^*} = 943.41 \pm 0.59 \text{ MeV}, \quad \gamma_{K^*} = 66.72 \pm 0.87 \text{ MeV}, \quad (21)$$

$$m_{K^{*'}} = 1374 \pm 45 \text{ MeV}, \quad \gamma_{K^{*'}} = 240 \pm 131 \text{ MeV}, \quad (22)$$

while the mixing parameter for the second resonance reads $\gamma = -0.039 \pm 0.020$. As has been already stated above, the quantities of eqs. (21) and (22) are unphysical fit parameters.

To obtain physical parameters for the K^* and $K^{*'}$ resonances, we have to compute the positions of the poles of the vector form factor in the complex s -plane. From the pole position s_p we can then read off the physical mass and width of the respective resonance according to the relation

$$\sqrt{s_p} = M_R - \frac{i}{2} \Gamma_R. \quad (23)$$

Calculating the pole positions along the lines of the approach outlined in ref. [49] yields:

$$M_{K^*} = 892.01 \pm 0.92 \text{ MeV}, \quad \Gamma_{K^*} = 46.20 \pm 0.38 \text{ MeV}, \quad (24)$$

$$M_{K^{*'}} = 1276_{-77}^{+72} \text{ MeV}, \quad \Gamma_{K^{*'}} = 198_{-87}^{+61} \text{ MeV}. \quad (25)$$

The uncertainties are calculated by assuming a Gaussian error propagation while simultaneously varying both m_R and γ_R . The mass of the charged K^* meson turns out rather close to the value advocated by the PDG, but more than 3 MeV lower than the Breit-Wigner resonance parameters obtained in refs. [24, 36]. On the other hand, the K^* width Γ_{K^*} of eq. (24) nicely agrees with the result of [24], but it is more than 4 MeV lower than the PDG average.

To shed further light on the comparison with previous works let us calculate the pole position of the $K^*(892)$ for the best fit of ref. [36]. The model employed in this reference amounts, as far as the poles are concerned, to removing the term proportional to $\text{Re}\tilde{H}_{K\pi}(s)$ from eq. (18) while keeping the energy dependent width. Denoting the respective fit parameters with a tilde we have $\tilde{m}_{K^*} = 895.28 \pm 0.20 \text{ MeV}$ and $\tilde{\gamma}_{K^*} = 47.50 \pm 0.41 \text{ MeV}$ [36]. The corresponding pole position is given as the second line in table 5, being perfectly consistent with our results provided in eq. (24) and the first line of table 5. The same exercise can be repeated for the original analysis performed by the Belle collaboration [24]. Here, we have employed the fit parameters corresponding to the second fit given in table 3, which are close to their final result for the K^* parameters. In this case the corresponding pole position is

displayed as the last line in table 5. Again it turns out rather close to the previous results. To summarise, the pole position is found to be rather stable since different models yield compatible values for the *physical* parameters M_{K^*} and Γ_{K^*} as defined in eq. (23). Concerning the parameters of the second resonance, they are in general agreement with the findings of ref. [36], especially after the pole position is computed for the latter results. Due to the large uncertainties, however, we cannot make any more definite statement.

	Model Parameters (m_{K^*}, γ_{K^*}) [MeV]	Pole Positions (M_{K^*}, Γ_{K^*}) [MeV]
This work	$(943.41 \pm 0.59, 66.72 \pm 0.87)$	$(892.0 \pm 0.9, 46.2 \pm 0.4)$
Ref. [36]	$(895.28 \pm 0.20, 47.50 \pm 0.41)$	$(892.1 \pm 0.2, 46.5 \pm 0.4)$
Ref. [24]	$(895.47 \pm 0.20, 46.19 \pm 0.57)$	$(892.5 \pm 0.2, 45.3 \pm 0.5)$

Table 5: Comparison between model parameters and corresponding pole positions for the charged $K^*(892)$ meson. For definiteness, from ref. [24] we have employed the parameters of the second fit of their table 3 and consider only statistical uncertainties.

Our fits to the $\tau^- \rightarrow \nu_\tau K_S \pi^-$ spectrum also allow to determine the slope parameters of the vector form factor $F_+^{K\pi}(s)$. The advantage of using a three-subtracted dispersion relation is that the parameters λ'_+ and λ''_+ are directly determined from the data, making the extraction more model independent. The disadvantage being that therefore the uncertainties for λ'_+ turn out larger than for example in ref. [36], where these parameters are a direct consequence of the form factor model. Higher slope parameters can of course also be calculated through dispersive integrals. For example in the case of λ'''_+ one has the relation:

$$\lambda'''_+ = \alpha_1^3 + 3\alpha_1\alpha_2 + M_{\pi^-}^6 \frac{6}{\pi} \int_{s_{K\pi}}^{s_{\text{cut}}} ds' \frac{\delta_1^{K\pi}(s')}{(s')^4}. \quad (26)$$

Together with the explicit fit results, this leads to

$$\lambda'_+ = (24.66 \pm 0.77) \cdot 10^{-3}, \quad \lambda''_+ = (11.99 \pm 0.20) \cdot 10^{-4}, \quad \lambda'''_+ = (8.73 \pm 0.16) \cdot 10^{-5}, \quad (27)$$

where again the uncertainty due to the variation of s_{cut} has been included in quadrature. Within the given errors, the value (27) for λ'_+ is in good agreement to the result of ref. [36], as well as the determination from an average of current experimental data for K_{l3} decays [47]. On the other hand, both, the quadratic slope λ''_+ , and the cubic slope λ'''_+ , are found somewhat lower than the corresponding results of ref. [36].

To conclude, differential decay spectra of hadronic τ decays provide important information for testing form factor models and extracting the corresponding model parameters, thereby accessing QCD in the realm of low energies. It will be very interesting to see if our findings are corroborated by additional experimental data in the future. Furthermore, when comparing parameters of hadronic resonances, even when employing Breit-Wigner type parametrisations

with an energy-dependent width, pole positions in the complex s -plane should be provided in order to arrive at more model independent results.

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